Exact Solutions for Shielded Printed Microstrip Lines by the Carleman–Vekua Method

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Abstract —**Exact analytical solutions for the field of the quasi-TEM mode in various cross-sectional configurations of rectangularly shielded printed microstrip lines are obtained on the basis of Carleman-type singular integral equations (SIE's). There are no limitations on the dimensions or the proximity of the strip conductors to the shield. For the kernel of the SIE, strongly and uniformly convergent series expansions have recently been developed that are suitable for the exact solution of the equation by the Carleman–Vekua regularization method, which proceeds by first solving the so-called dominant equation. The procedure leads to rapidly convergent series solutions for the field of the quasi-TEM mode even when the conductors are large or very near the shield, i.e., in situations for which numerical techniques become inadequate. Characteristic values of the shielded microstrip lines are evaluated by summing a few terms, while field plots, requiring more terms, are shown for various configurations including the case of close proximity.**

I. Introduction

THIS PAPER is based on the results of a previous paper [1] by the authors, in which rapidly convergent Green's function expansions for rectangularly shielded printed microstrip lines were developed. The main objectives and the rationale of the method are fully explained in [1] and in two other recent papers by the first two of the authors [2], [3] and will not be repeated. We merely stress here what we believe to be a unique advantage of our analytical approach: It is able to provide exact results for the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields of the mode at any point inside the guide, in particular, near edges or when the conductors are large or close to each other or to the walls. In corresponding scattering problems by, for example, strips on substrates— to which our approach can be further extended —it provides accurate near-field evaluations. These are situations where well-known numerical techniques (Galerkin, finite elements, etc.) prove inaccurate, as discussed further in [1] and [3]. Such techniques are sufficient and preferable for quantities like the characteristic impedance Z_0 or the effective dielectric constant of the line, owing to the stationary character of the related integral formula; even then, however, cases of close proximity are excluded. Furthermore, in our opinion, they often fail to identify the main influence of various factors affecting the solution, something important to research and design.

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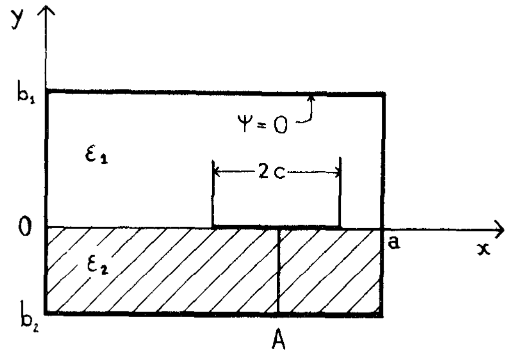


Fig. 1. Configuration of shielded single microstrip line.

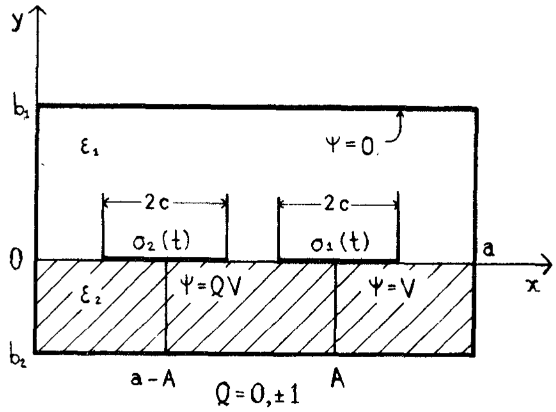


Fig. 2. Configuration of shielded double microstrip line.

Our method, on the other hand, systematically examines their influence (particularly on the Green's function of the problem) and provides alternative, fast-converging evaluation procedures and a clear estimate of their accuracy [1]–[3]. Finally, we remark that the quasi-TEM mode is used in the literature as a zeroth-order solution in an iterative evaluation of the true lower hybrid mode of the line.

The microstrip line configurations are shown in Figs. 1 and 2. Two dielectric sublayers ϵ_1 and ϵ_2 divided by the surface $y=0$ are enclosed in a rectangular shield $a\times(b_1-b_2)$. The strip conductors of width $2c$ are printed on the interface with center at $x=A$ for one-conductor configurations (Fig. 1) and at A and $a-A$ for symmetric two-conductor ones (Fig. 2).

The Green's function G for these configurations is a potential function with zero value on the shield surface and a unit line source on the interface at $(x',0)$. It is so defined that the electrostatic potential $\psi(x, y)$ of a line charge q (C/m) at $(x',0)$ at any point (x, y) inside the shield is: $\psi(x, y) = qG(x, y; x')/\pi(\epsilon_1 + \epsilon_2)$. For G four expansions have been developed in [1]. The last, G_δ, is the more appropriate one when line sources are near the walls $x = 0, a$ and when $y \cong 0$; this last condition, $y = 0$, is always imposed in the process of obtaining the integral equation for strip conductors printed on $y = 0$. We therefore provide here the explicit expansion G_δ for G. With $k = 1,2$ denoting regions $\epsilon_1(0 \leqslant y \leqslant b_1)$, $\epsilon_2(b_2 \leqslant y \leqslant 0)$ in the cross-sectional configuration of the line, we have from [1]

$$G_\delta = G_k(x, y; x') = -\frac{1}{2}\ln\left[(x-x')^2 + y^2\right] + G_k^c(x, y; x'), \qquad k = 1,2 \tag{1}$$

$$G_k^c = \frac{1}{2}\ln\left[(x+x'-2a)^2 + y^2\right] + \frac{1}{2}\ln\left[(x+x')^2 + y^2\right] - \frac{1}{ab_k}\Big\{(a-x)(b_k - y)\ln(2a-x')$$
$$+ x(b_k - y)\ln(a+x') + \frac{1}{2}(a-x)y\ln\left[(2a-x')^2 + b_k^2\right] + \frac{xy}{2}\ln\left[(a+x')^2 + b_k^2\right]\Big\} + \sum_{j=1}^{4} S_j'^{(k)}(x, y; x') \tag{2}$$

$$S_1'^{(k)}(x, y; x') = \sum_{m=1}^{\infty} \frac{\sin\left(\frac{m\pi x}{a}\right)\sinh\left(\frac{m\pi y}{a}\right)}{m\pi \sinh\left(\frac{m\pi}{a}b_k\right)}\Big\{-2\pi\sin\left(\frac{m\pi x'}{a}\right)\exp\left(-\frac{m\pi}{a}|b_k|\right)$$
$$- d_m\left[\frac{\pi}{a}(-b_k + ix')\right] + d_m\left[\frac{\pi}{a}(-b_k - ix')\right] + d_m\left[\frac{\pi}{a}(-b_k - i(x'-2a))\right]\Big\} \tag{3}$$

$$S_3'^{(k)}(x, y; x') = \sum_{m=1}^{\infty} \frac{\sin\left(\frac{m\pi y}{b_k}\right)\sinh\left(\frac{m\pi x}{b_k}\right)}{m\pi\sinh\left(\frac{m\pi a}{b_k}\right)} d_m\left(\pi\frac{-a-x'}{b_k}\right) \tag{4}$$

$$S_4'^{(k)}(x, y; x') = \sum_{m=1}^{\infty} \frac{\sin\left(\frac{m\pi y}{b_k}\right)\sinh\left(m\pi\frac{a-x}{b_k}\right)}{m\pi\sinh\left(\frac{m\pi a}{b_k}\right)} d_m\left(\pi\frac{-2a+x'}{b_k}\right) \tag{5}$$

$$S_2'^{(k)}(x, y; x') = -\sum_{m=1}^{\infty}\left\{\sin\left(\frac{m\pi x}{a}\right)\sinh\left[\frac{m\pi}{a}(b_k - y)\right]\Big/\left[m\pi\sinh\left(\frac{m\pi b_k}{a}\right)B_m\right]\right\}\sum_{l=1,2} D_{ml}(x')$$
$$-(1 - y/b_k)\left[g(a+x')x - g(2a-x')(x-a)\right]\Big/\left[a\left(\frac{\epsilon_1}{b_1} - \frac{\epsilon_2}{b_2}\right)\right]$$
$$+2b_1^2\sum_{n=1}^{\infty}\sin\left(\frac{b_k - y}{b_1}u_n\right)\left[\sinh\left(\frac{u_n x}{b_1}\right)g(a+x') + \sinh\left(\frac{a-x}{b_1}u_n\right)g(2a-x')\right]\Big/\left[u_n^2\sin\left(\frac{u_n b_k}{b_1}\right)H_n\right]$$
$$+\sum_{l=1,2}(-1)^l\epsilon_l\sum_{q=1}^{\infty}\left\{\frac{\sin\left(\frac{\pi q y}{b_l}\right)\delta\left(\frac{b_k}{b_l}q\right)\left[d_q\left(\pi\frac{x'-2a}{b_l}\right)\sinh\left(\pi q\frac{a-x}{b_l}\right) + d_q\left(\pi\frac{-a-x'}{b_l}\right)\sinh\left(\frac{\pi q x}{b_l}\right)\right]}{q\pi b_k\sinh\left(\frac{\pi q a}{b_l}\right)\left[\frac{\epsilon_1}{b_1}\delta\left(\frac{b_1}{b_l}q\right) - \frac{\epsilon_2}{b_2}\delta\left(\frac{b_2}{b_l}q\right)\right]}\right.$$
$$-\frac{2}{b_l}\sum_{n=1}^{\infty}\sin\left(u_n\frac{b_k - y}{b_1}\right)\left[d_q\left(\pi\frac{x'-2a}{b_l}\right)\sinh\left(u_n\frac{a-x}{b_1}\right)\right.$$
$$\left.\left.+ d_q\left(\pi\frac{-a-x'}{b_l}\right)\sinh\left(u_n\frac{x}{b_1}\right)\right]\Big/\left\{H_n\sin\left(u_n\frac{b_k}{b_1}\right)\left[\left(\frac{q\pi}{b_l}\right)^2 - \left(\frac{u_n}{b_1}\right)^2\right]\right\}\right\} \tag{6}$$

where, with $\bar{z}$ denoting the complex conjugate of z,

$$d_m(z) = \mathrm{Re}\{\exp(mz)[E_1(mz - im\pi) - E_1(mz)] + \exp(-m\bar{z})[E_1(-m\bar{z} - im\pi) - E_1(-m\bar{z})]\} = d_m(-\bar{z}) \quad (7)$$

$E_1(z)$ being the exponential integral function [1], [4]. Also,

$$B_m = \epsilon_1 \coth\left(\frac{m\pi}{a}b_1\right) - \epsilon_2 \coth\left(\frac{m\pi}{a}b_2\right) \quad (8)$$

$$H_n = \sinh\left(\frac{a}{b_1}u_n\right)\left[\frac{\epsilon_1 b_1}{\sin^2 u_n} - \frac{\epsilon_2 b_2}{\sin^2(u_n b_2/b_1)}\right] \quad (9)$$

$$D_{ml}(x') = (-1)^{l+1}\epsilon_l\left\{2\pi\exp\left(-\frac{m\pi}{a}|b_l|\right)\sin\left(\frac{m\pi x'}{a}\right) + d_m\left[\frac{\pi}{a}(-b_l + ix')\right] - d_m\left[\frac{\pi}{a}(-b_l - ix')\right] - d_m\left[\frac{\pi}{a}(-b_l - i(x' - 2a))\right]\right\}\Big/\sinh\left(\frac{m\pi}{a}b_l\right) \quad (10)$$

$$g(x') = \frac{\epsilon_1}{2b_1}\ln(x'^2 + b_1^2) - \frac{\epsilon_2}{2b_2}\ln(x'^2 + b_2^2) - \left(\frac{\epsilon_1}{b_1} - \frac{\epsilon_2}{b_2}\right)\ln|x'|. \quad (11)$$

Also, we define $\delta(x) = 1$ for $x =$ integer, $\delta(x) = 0$ otherwise, while u_n are the positive roots of the transcendental equation

$$\epsilon_1 \cot u = \epsilon_2 \cot\left(\frac{b_2}{b_1}u\right). \quad (12)$$

Obviously, in this form of G three logarithmic terms are extracted out in closed form, corresponding to line sources at $(x',0)$ and at (the image positions) $(-x',0)$ and $(2a - x',0)$.

Now let $\sigma_1(x)$ (C/m^2) be the surface charge distribution of the single-strip conductor of Fig. 1. The electrostatic potential function at any point (x, y) inside the shield is

$$\psi_k(x, y) = \frac{1}{\pi(\epsilon_1 + \epsilon_2)}\int_{A-c}^{A+c}\sigma_1(x')G_k(x, y; x')\,dx'. \quad (13)$$

If the potential of the strip conductor is $\psi(x,0) = V$ $(A - c \leqslant x \leqslant A + c)$, a Carleman-type singular integral equation (SIE) [5], [6] is obtained for $\sigma_1(x)$ by letting $y = 0$ in (13). Observing further that $G_k^c(x,0;x')$ is the same for either $k = 1$ or $k = 2$, owing to the continuity condition at $y = 0$, and setting $x = A + ct$, $x' = A + ct'$ $(-1 \leqslant t, t' \leqslant 1)$, and

$$\sigma_1(A + ct') = \sigma(t'), \qquad G_k^c(A + ct, 0; A + ct') = G^c(t, t') \quad (14)$$

we end up with the "traditional" form of the Carleman-type equation:

$$-\frac{\pi(\epsilon_1 + \epsilon_2)V}{c} - \ln c\int_{-1}^{1}\sigma(t')\,dt' + \int_{-1}^{1}G^c(t,t')\sigma(t')\,dt' = \int_{-1}^{1}\ln|t - t'|\sigma(t')\,dt'. \quad (15)$$

It should be noted that for $y = 0$ the expression $S_2'^{(k)}(x,0;x')$ simplifies considerably while $S_j'^{(k)}(x,0;x') = 0$ for $j = 1,3,4$.

II. Solution of the Integral Equation by the Carleman–Vekua Method

If the left-hand side of (15) is considered for a moment to be a known function of t, then the Carleman–Vekua method of regularization [6] proceeds by inverting it with the use of Carleman's formula [5]:

$$\sigma(t) = \frac{1}{\pi^2\sqrt{1-t^2}}\left\{⨍_{-1}^{1}\frac{\sqrt{1-\tau^2}}{\tau - t}\int_{-1}^{1}\sigma(t')\frac{\partial G^c(\tau,t')}{\partial\tau}\,d\tau\,dt' - \frac{1}{\ln 2}\int_{-1}^{1}\left[-\frac{\epsilon_1 + \epsilon_2}{c}\pi V - \ln c\int_{-1}^{1}\sigma(t')\,dt' + \int_{-1}^{1}\sigma(t')G^c(\tau,t')\,dt'\right]\frac{d\tau}{\sqrt{1-\tau^2}}\right\} \quad (16)$$

where $⨍$ denotes principal value integral. This equation may be written as a Fredholm-type integral equation [6] in the concise form

$$\sigma(t) + \int_{-1}^{1}K(t,t')\sigma(t')\,dt' = h(t)$$
$$h(t) = \frac{1}{\pi\sqrt{1-t^2}}\frac{1}{\ln 2}\left[\frac{\epsilon_1 + \epsilon_2}{c}\pi V + \ln c\int_{-1}^{1}\sigma(t')\,dt'\right] \quad (17)$$

Here use of the simple result $\int_{-1}^{1}d\tau/\sqrt{1-\tau^2} = \pi$ was made. Also,

$$K(t,t') = \frac{1}{\pi^2\sqrt{1-t^2}}\left[\frac{1}{\ln 2}\int_{-1}^{1}G^c(\tau,t')\frac{d\tau}{\sqrt{1-\tau^2}} - ⨍_{-1}^{1}\frac{\sqrt{1-\tau^2}}{\tau - t}\frac{\partial G^c(\tau,t')}{\partial\tau}\,d\tau\right]$$
$$= \frac{1}{\pi^2\sqrt{1-t^2}}\left[\frac{1}{\ln 2}\int_{-1}^{1}\left\{\ln|2a - 2A - c(\tau + t')| + \ln|2A + c(\tau + t')|\right.\right.$$
$$-\left(1 - \frac{A}{a} - \frac{c}{a}\tau\right)\ln|2a - A - ct'| - \left(\frac{A}{a} + \frac{c}{a}\tau\right)\ln|a + A + ct'|$$
$$-\left[\left(\frac{A}{a} + \frac{c}{a}\tau\right)g(a + A + ct') - \left(\frac{A}{a} - 1 + \frac{c}{a}\tau\right)g(2a - A - ct')\right]\Big/\left(\frac{\epsilon_1}{b_1} - \frac{\epsilon_2}{b_2}\right)$$

$$
- \sum_{m=1}^{\infty} \left[\sin\left(m\pi \frac{A+c\tau}{a} \right) \Big/ (m\pi B_m) \right] \sum_{l=1,2} D_{ml}(A+ct')
$$

$$
+2b_1^2 \sum_{n=1}^{\infty} \left[\sinh\left(u_n \frac{A+c\tau}{b_1} \right) g(a+A+ct') + \sinh\left(u_n \frac{a-A-c\tau}{b_1} \right) g(2a-A-ct') \right] \Big/ \left(u_n^2 H_n \right)
$$

$$
-2 \sum_{l=1,2} (-1)^l \frac{\epsilon_l}{b_l} \sum_{n=1}^{\infty} \sum_{q=1}^{\infty} \left[d_q\left(\pi \frac{A-2a+ct'}{b_l} \right) \sinh\left(u_n \frac{a-A-c\tau}{b_1} \right) + d_q\left(\pi \frac{-a-A-ct'}{b_l} \right) \right.
$$

$$
\left. \cdot \sinh\left(u_n \frac{A+c\tau}{b_1} \right) \right] \Big/ \left\{ H_n \left[\left(\frac{q\pi}{b_l} \right)^2 - \left(\frac{u_n}{b_1} \right)^2 \right] \right\} \Bigg\} \frac{d\tau}{\sqrt{1-\tau^2}} - \int_{-1}^{1} \frac{\sqrt{1-\tau^2}}{\tau - t} \left\{ \frac{c}{2A+c(\tau+t')} \right.
$$

$$
- \frac{c}{2a-2A-c(\tau+t')} + \frac{c}{a} \ln|2a-A-ct'| - \frac{c}{a} \ln|a+A+ct'| - \frac{c}{a} [g(a+A+ct')
$$

$$
- g(2a-A-ct')] \Big/ \left(\frac{\epsilon_1}{b_1} - \frac{\epsilon_2}{b_2} \right) - \frac{c}{a} \sum_{m=1}^{\infty} \frac{1}{B_m} \cos\left(m\pi \frac{A+c\tau}{a} \right) \sum_{l=1,2} D_{ml}(A+ct')
$$

$$
+2b_1 c \sum_{n=1}^{\infty} \left[\cosh\left(u_n \frac{A+c\tau}{b_1} \right) g(a+A+ct') - \cosh\left(u_n \frac{a-A-c\tau}{b_1} \right) g(2a-A-ct') \right] \Big/ (u_n H_n)
$$

$$
-2 \frac{c}{b_1} \sum_{l=1,2} (-1)^l \frac{\epsilon_l}{b_l} \sum_{n=1}^{\infty} u_n \sum_{q=1}^{\infty} \left[d_q\left(\pi \frac{-a-A-ct'}{b_l} \right) \cosh\left(u_n \frac{A+c\tau}{b_1} \right) - d_q\left(\pi \frac{A-2a+ct'}{b_l} \right) \right.
$$

$$
\left. \cdot \cosh\left(u_n \frac{a-A-c\tau}{b_1} \right) \right] \Big/ \left\{ H_n \left[\left(\frac{q\pi}{b_l} \right)^2 - \left(\frac{u_n}{b_1} \right)^2 \right] \right\} \Bigg\} d\tau \Bigg]. \qquad (18)
$$

The step from (16) to (17) and (18) involves a change in the order of integration over t' and τ, both for the ordinary and the principal value integral. Since $G^c(\tau, t)$ and $\partial G^c(\tau, t)/\partial \tau$ are nonsingular functions in $-1 \leq t', \tau \leq 1$, as seen from (1) and (2)–(6), this change is permissible [6, pp. 47–52].

We next expand $\sigma(t)$ in a series of Chebyshev polynomials:

$$
\sigma(t) = \frac{1}{\sqrt{1-t^2}} \sum_{N=0}^{\infty} a_N T_N(t), \qquad -1 \leq t \leq 1 \qquad (19)
$$

incorporating, with the common factor $(1-t^2)^{-1/2}$, already present in $h(t)$ and $K(t,t')$, its expected behavior at the edges $t = \pm 1$. Substituting into (17), multiplying by $T_M(t)$ $(M = 0,1,2,\cdots)$, integrating from $t=-1$ to $t=1$, and using the orthogonal property

$$
\int_{-1}^{1} T_N(t) T_M(t) \frac{dt}{\sqrt{1-t^2}} = \delta_{NM} \epsilon_M \pi/2
$$

$$
\delta_{NM} = \begin{cases} 0, & N \neq M \\ 1, & N = M \end{cases} \qquad \epsilon_M = \begin{cases} 1, & M > 0 \\ 2, & M = 0 \end{cases} \qquad (20)
$$

we end up with the following set of linear equations:

$$
a_M \frac{\pi}{2} \epsilon_M + \sum_{N=0}^{\infty} K_{MN} a_N = \frac{\delta_{M0}\pi}{\ln 2} \left(\frac{\epsilon_1 + \epsilon_2}{c} V + a_0 \ln c \right),
$$

$$
M, N = 0,1,2,\cdots \qquad (21)
$$

in which

$$
K_{MN} = \int_{-1}^{1} \frac{T_N(t')}{\sqrt{1-t'^2}} \int_{-1}^{1} T_M(t) K(t,t')\, dt\, dt'. \qquad (22)
$$

As seen from (18) the coefficients K_{MN} require the evaluation of triple integrals over t, t', τ. The advantage is that the dependence of $K(t,t')$ on t, t' appears in separated form and the series involved converge rapidly and uniformly. The order of integration can be interchanged in any desired way, greatly facilitating the evaluation of K_{MN}. The result can be expressed in terms of six integrals, I_1 to I_6, defined below in the order in which they come up in expressions (18) and (22). These integrals are evaluated in the Appendix. Two preliminary results help reduce the number of symbols. From (20) and the first integral in the Appendix, we have

$$
\int_{-1}^{1} \frac{T_N(t)(z_1 + z_2 t)}{\sqrt{1-t^2}} dt = z_1 \pi \delta_{N0} + z_2 \frac{\pi}{2} \delta_{N1} \qquad (23)
$$

$$
I = \int_{-1}^{1} \frac{T_M(t)}{\sqrt{1-t^2}} \int_{-1}^{1} \frac{\sqrt{1-\tau^2}}{\tau - t} d\tau\, dt
$$

$$
= -\frac{\pi^2}{2} \delta_{M1}. \qquad (24)
$$

The triple integrals appearing in (18) and (22) are then

$$\int_{-1}^{1}\frac{T_N(t')}{\sqrt{1-t'^2}}\int_{-1}^{1}\frac{T_M(t)}{\sqrt{1-t^2}}\int_{-1}^{1}\frac{\ln|z_1+z_2(\tau+t')|}{\sqrt{1-\tau^2}}\,dt'dt\,d\tau=\pi\delta_{M0}I_1(N;z_1,z_2) \tag{25}$$

$$I_1(N;z_1,z_2)=\int_{-1}^{1}\frac{T_N(t')}{\sqrt{1-t'^2}}\int_{-1}^{1}\frac{\ln|z_1+z_2(\tau+t')|}{\sqrt{1-\tau^2}}\,dt'\,d\tau,\qquad |z_2/z_1|<1/2 \tag{26}$$

$$\int_{-1}^{1}\frac{T_N(t')\ln|z_1+z_2t'|}{\sqrt{1-t'^2}}\int_{-1}^{1}\frac{T_M(t)}{\sqrt{1-t^2}}\int_{-1}^{1}\frac{z_3+z_4\tau}{\sqrt{1-\tau^2}}\,dt'\,dt\,d\tau=\frac{\pi^2}{2}\delta_{M0}z_3I_2(N;z_1,z_2,0) \tag{27}$$

$$\int_{-1}^{1}\frac{T_N(t')\ln\left[(z_1+z_2t')^2+z_3^2\right]}{\sqrt{1-t'^2}}\int_{-1}^{1}\frac{T_M(t)}{\sqrt{1-t^2}}\int_{-1}^{1}\frac{z_4+z_5\tau}{\sqrt{1-\tau^2}}\,dt'\,dt\,d\tau$$
$$=\pi^2\delta_{M0}z_4I_2(N;z_1,z_2,z_3) \tag{28}$$

$$I_2(N;z_1,z_2,z_3)=\int_{-1}^{1}\frac{T_N(t')\ln\left[(z_1+z_2t')^2+z_3^2\right]}{\sqrt{1-t'^2}}\,dt',\qquad -1<\frac{z_2}{z_1}<1,\quad z_3\geqslant 0 \tag{29}$$

$$\int_{-1}^{1}\frac{T_N(t')\sin(z_1+z_2t')}{\sqrt{1-t'^2}}\int_{-1}^{1}\frac{T_M(t)}{\sqrt{1-t^2}}\int_{-1}^{1}\frac{\sin(z_3+z_4\tau)}{\sqrt{1-\tau^2}}\,dt'dt\,d\tau=\pi\delta_{M0}I_3(N;z_1,z_2)I_3(0;z_3,z_4) \tag{30}$$

$$I_3(N;z_1,z_2)=\int_{-1}^{1}\frac{T_N(t')\sin(z_1+z_2t')}{\sqrt{1-t'^2}}\,dt' \tag{31}$$

$$\int_{-1}^{1}\frac{T_N(t')d_m(z_1+z_2t')}{\sqrt{1-t'^2}}\int_{-1}^{1}\frac{T_M(t)}{\sqrt{1-t^2}}\int_{-1}^{1}\frac{\sin(z_3+z_4\tau)}{\sqrt{1-\tau^2}}\,dt'dt\,d\tau=\pi\delta_{M0}I_3(0;z_3,z_4)I_4(N,m;z_1,z_2) \tag{32}$$

$$I_4(N,m;z_1,z_2)=\int_{-1}^{1}\frac{T_N(t')\,d_m(z_1+z_2t')}{\sqrt{1-t'^2}}\,dt' \tag{33}$$

$$\int_{-1}^{1}\frac{T_N(t')g(z_1+z_2t')}{\sqrt{1-t'^2}}\int_{-1}^{1}\frac{T_M(t)}{\sqrt{1-t^2}}\int_{-1}^{1}\frac{\sinh(z_3+z_4\tau)}{\sqrt{1-\tau^2}}\,dt'\,dt\,d\tau=-i\pi\delta_{M0}I_3(0;iz_3,iz_4)g(N;z_1,z_2) \tag{34}$$

$$g(N;z_1,z_2)=\frac{1}{2}\left[\frac{\epsilon_1}{b_1}I_2(N;z_1,z_2,b_1)-\frac{\epsilon_2}{b_2}I_2(N;z_1,z_2,b_2)-\left(\frac{\epsilon_1}{b_1}-\frac{\epsilon_2}{b_2}\right)I_2(N;z_1,z_2,0)\right] \tag{35}$$

$$I_5(N,M;z_1,z_2)=\int_{-1}^{1}\frac{T_N(t')}{\sqrt{1-t'^2}}\int_{-1}^{1}\frac{T_M(t)}{\sqrt{1-t^2}}\fint_{-1}^{1}\frac{\sqrt{1-\tau^2}}{\tau-t}\,\frac{d\tau}{z_1+z_2(\tau+t')}\,dt'\,dt,\qquad \left|\frac{z_2}{z_1}\right|<\frac{1}{2} \tag{36}$$

$$\int_{-1}^{1}\frac{T_N(t')\ln|z_1+z_2t'|}{\sqrt{1-t'^2}}\int_{-1}^{1}\frac{T_M(t)}{\sqrt{1-t^2}}\fint_{-1}^{1}\frac{\sqrt{1-\tau^2}}{\tau-t}\,dt'\,dt\,d\tau=-\frac{\pi^2}{4}\delta_{M1}I_2(N;z_1,z_2,0) \tag{37}$$

$$\int_{-1}^{1}\frac{T_N(t')\sin(z_1+z_2t')}{\sqrt{1-t'^2}}\int_{-1}^{1}\frac{T_M(t)}{\sqrt{1-t^2}}\int_{-1}^{1}\frac{\sqrt{1-\tau^2}\cos(z_3+z_4\tau)}{\tau-t}\,dt'\,dt\,d\tau$$
$$=I_3(N;z_1,z_2)I_6(M;z_3,z_4) \tag{38}$$

$$I_6(M;z_3,z_4)=\int_{-1}^{1}\frac{T_M(t)}{\sqrt{1-t^2}}\fint_{-1}^{1}\frac{\sqrt{1-\tau^2}\cos(z_3+z_4\tau)}{\tau-t}\,dt\,d\tau \tag{39}$$

$$\int_{-1}^{1}\frac{T_N(t')d_m(z_1+z_2t')}{\sqrt{1-t'^2}}\int_{-1}^{1}\frac{T_M(t)}{\sqrt{1-t^2}}\fint_{-1}^{1}\frac{\sqrt{1-\tau^2}\cos(z_3+z_4\tau)}{\tau-t}\,dt'\,dt\,d\tau$$
$$=I_4(N,m;z_1,z_2)I_6(M;z_3,z_4) \tag{40}$$

$$\int_{-1}^{1}\frac{T_N(t')g(z_1+z_2t')}{\sqrt{1-t'^2}}\int_{-1}^{1}\frac{T_M(t)}{\sqrt{1-t^2}}\fint_{-1}^{1}\frac{\sqrt{1-\tau^2}\cosh(z_3+z_4\tau)}{\tau-t}\,dt'\,dt\,d\tau$$
$$=I_6(M;iz_3,iz_4)g(N;z_1,z_2) \tag{41}$$

$$\int_{-1}^{1}\frac{T_N(t')d_q(z_1+z_2t')}{\sqrt{1-t'^2}}\int_{-1}^{1}\frac{T_M(t)}{\sqrt{1-t^2}}\int_{-1}^{1}\frac{\sqrt{1-\tau^2}\cosh(z_3+z_4\tau)}{\tau-t}\,dt'\,dt\,d\tau$$
$$= I_4(N,q;z_1,z_2)I_6(M;iz_3,iz_4). \tag{42}$$

The final expression for K_{MN} can now be written in a concise expression by introducing the two additional symbols $L_{MN}(A,c)$ and $F_{MN}(A,c)$. Thus,

$$L_{MN}(A,c) = -\left(\frac{A}{a}\frac{\delta_{M0}}{\ln 2}+\frac{c}{2a}\delta_{M1}\right)\left[\frac{1}{2}I_2(N;a+A,c,0)+g(N;a+A,c)\Big/\left(\frac{\epsilon_1}{b_1}-\frac{\epsilon_2}{b_2}\right)\right]$$
$$+\frac{\delta_{M0}}{\pi\ln 2}I_1(N;2A,c)-\frac{c}{\pi^2}I_5(N,M;2A,c) \tag{43}$$

$$F_{MN}(A,c) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{i\delta_{M0}}{u_n\ln 2}I_3\left(0;i\frac{u_nA}{b_1},i\frac{u_nc}{b_1}\right)+\frac{c}{\pi b_1}I_6\left(M;i\frac{u_nA}{b_1},i\frac{u_nc}{b_1}\right)\right]$$
$$\cdot\frac{1}{H_n}\left\{\frac{b_1^2}{u_n}g(N;a+A,c)-u_n\sum_{l=1,2}(-1)^l\frac{\epsilon_l}{b_l}\sum_{q=1}^{\infty}I_4\left[N,q;-\frac{\pi}{b_l}(a+A),-\frac{\pi c}{b_l}\right]\Big/\left[\left(\frac{q\pi}{b_l}\right)^2-\left(\frac{u_n}{b_1}\right)^2\right]\right\} \tag{44}$$

$$K_{MN} = L_{MN}(A,c)+L_{MN}(a-A,-c)+F_{MN}(A,c)+F_{MN}(a-A,-c)-\frac{1}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{B_m}$$
$$\cdot\left[-\frac{\delta_{M0}}{m\ln 2}I_3\left(0;\frac{m\pi A}{a},\frac{m\pi c}{a}\right)+\frac{c}{a}I_6\left(M;\frac{m\pi A}{a},\frac{m\pi c}{a}\right)\right]\sum_{l=1,2}D_{mNl} \tag{45a}$$

$$D_{mNl} = \frac{(-1)^l\epsilon_l}{\sinh\left(\frac{m\pi}{a}b_l\right)}\left[2\pi\exp\left(-\frac{m\pi}{a}|b_l|\right)I_3\left(N;\frac{m\pi A}{a},\frac{m\pi c}{a}\right)+I_4\left(N,m;-\frac{\pi}{a}b_l+i\frac{\pi}{a}A,i\frac{\pi c}{a}\right)\right.$$
$$\left.-I_4\left(N,m;-\frac{\pi}{a}b_l-i\frac{\pi}{a}A,-i\frac{\pi c}{a}\right)-I_4\left(N,m;-\frac{\pi}{a}b_l-i\frac{\pi}{a}(A-2a),-i\frac{\pi c}{a}\right)\right]. \tag{45b}$$

III. Double-Strip Configurations

Such configurations lead to a system of integral equations for $\sigma_1(t_1)$ and $\sigma_2(t_2)$. More practical cases consist of strips with the same width, $2c$, and centers at $x=A$ and $x=a-A$, i.e., symmetrically placed with respect to the midplane $x=a/2$, as shown in Fig. 2. They are raised to potentials V and QV, with $Q=1$ implying equal currents and charges, $Q=-1$ opposite ones, and $Q=0$ the absence of the left strip, i.e., the single-strip configuration of Fig. 1. Owing to the symmetry

$$\sigma_2(-t)=Q\sigma_1(t)=Q\sigma(t), \qquad -1\leqslant t\leqslant 1 \tag{46}$$

and the potential $\psi_k(x,y)$ at any point inside the shield is

$$\psi_k(x,y)=\frac{1}{\pi(\epsilon_1+\epsilon_2)}\int_{A-c}^{A+c}\sigma_1(x')G_k(x,y;x')\,dx'$$
$$+\int_{a-A-c}^{a-A+c}\sigma_2(x'')G_k(x,y;x'')\,dx'', \qquad k=1,2. \tag{47}$$

Letting $y=0$, $x=A+ct$, $x'=A+ct'$, $x''=a-A+ct''$, $G_k^c(A+ct,0;A+ct')=G^c(t,t')$, and $G_k^c(A+ct,0;a-A+ct'')=G_2^c(t,t'')$, with t,t',t'' in $[-1,1]$, we obtain a Carleman-type singular integral equation for $\sigma(t)$:

$$-\frac{\pi}{c}(\epsilon_1+\epsilon_2)V-\ln c\int_{-1}^{1}\sigma(t')\,dt'+\int_{-1}^{1}\sigma(t')G^c(t,t')\,dt'$$
$$-Q\ln|2A-a|\int_{-1}^{1}\sigma(-t'')\,dt''+Q\int_{-1}^{1}\sigma(-t'')$$
$$\cdot\left[G_2^c(t,t'')-\ln\left|1+\frac{c}{2A-a}(t-t'')\right|\right]dt''$$
$$=\int_{-1}^{1}\sigma(t')\ln|t-t'|\,dt'. \tag{48}$$

Inverting this as before with the use of Carleman's formula [5], we end up with a relation similar to (17), namely,

$$\sigma(t)+\int_{-1}^{1}K(t,t')\sigma(t')\,dt'$$
$$+Q\int_{-1}^{1}K_2(t,t')\,\sigma(-t')\,dt'=h_2(t) \tag{49}$$

where

$$h_2(t) = \frac{1}{\pi \ln 2}\frac{1}{\sqrt{1-t^2}}\left[\frac{\pi}{c}(\epsilon_1+\epsilon_2)V + \ln c\int_{-1}^{1}\sigma(t')\,dt' + Q\ln|2A-a|\int_{-1}^{1}\sigma(-t')\,dt'\right] \quad (50)$$

$$K_2(t,t') = \frac{1}{\pi^2\sqrt{1-t^2}}\left\{\frac{1}{\ln 2}\int_{-1}^{1}\left[G_2^c(\tau,t') - \ln\left|1+\frac{c}{2A-a}(\tau-t')\right|\right]\frac{d\tau}{\sqrt{1-\tau^2}} - \int_{-1}^{1}\frac{\sqrt{1-\tau^2}}{\tau-t}\right.$$

$$\left.\cdot\left[\frac{\partial G_2^c(\tau,t')}{\partial\tau} - \frac{c/(2A-a)}{1+\frac{c}{2A-a}(\tau-t')}\right]d\tau\right\}. \quad (51)$$

It is worth noting here that $G_2^c(\tau,t')$ differs from $G^c(\tau,t')$ only in that $x'=A+ct'$ in the former is replaced by $a-A+ct'$ in the latter, a change that can be easily traced down the sequence of equations (2), (6), (10), (11), (18). To save space we will not give for $K_2(t,t')$ a result as explicit as (18) for $K(t,t')$, but will proceed to the solution of (49) through the substitution (19) for $\sigma(t)$ and the sequence of steps following it up to (21) and (22). We will then provide explicit results for the new coefficients $K_{MN}(Q)$, where $K_{MN}(0)=K_{MN}$ as defined in (22) and (45). Two additional integrals, which are also evaluated in the Appendix, should first be defined:

$$P_{MN} = -\frac{1}{\pi^2\ln 2}\int_{-1}^{1}\frac{T_N(t')}{\sqrt{1-t'^2}}\int_{-1}^{1}\frac{T_M(t)}{\sqrt{1-t^2}}$$

$$\int_{-1}^{1}\ln\left|1+\frac{c}{2A-a}(\tau-t')\right|\frac{d\tau}{\sqrt{1-\tau^2}}\,dt'\,dt \quad (52)$$

$$W_{MN} = \frac{c}{\pi^2(2A-a)}\int_{-1}^{1}\frac{T_N(t')}{\sqrt{1-t'^2}}$$

$$\cdot\int_{-1}^{1}\frac{T_M(t)}{\sqrt{1-t^2}}\int_{-1}^{1}\frac{\sqrt{1-\tau^2}}{\tau-t}$$

$$\cdot\frac{d\tau}{1+\frac{c}{2A-a}(\tau-t')}\,dt'\,dt. \quad (53)$$

In place of (21) we now end up with the following set of linear equations for the determination of the expansion coefficients a_N of $\sigma(t)$ in (19):

$$a_M\frac{\pi}{2}\epsilon_M + \sum_{N=0}^{\infty}K_{MN}(Q)a_N$$

$$= \frac{\delta_{M0}\pi}{\ln 2}\left[\frac{(\epsilon_1+\epsilon_2)V}{c} + a_0(\ln c + Q\ln|2A-a|)\right],$$

$$M,N = 0,1,2,\cdots \quad (54)$$

where

$$K_{MN}(Q) = K_{MN} + Q(-1)^N(P_{MN}+W_{MN}+K'_{MN}),$$

$$Q = 0, \pm 1. \quad (55)$$

K'_{MN} are obtained from K_{MN}, given in (43)–(45), if the following substitutions are made in these equations: The integrals $I_3(0; z_1, z_2)$ (for $N=0$) and I_6 remain exactly the same; in the integrals $I_3(N; z_1, z_2)$ (for $N>0$), I_2, and I_4, as well as in the expression $g(N; z_1, z_2)$ defined in terms of I_2 in (35), A is replaced by $a-A$; in the integrals I_1 and I_5 the parameter $2A$ is replaced by a. Otherwise expressions (43)–(45) remain unchanged.

IV. Evaluation of the Field

The potential $\psi_k(x,y)$ at any point inside the shield is given by (13) for the single-strip configuration and by (47) for the double strip-configuration. Changing the variables $x'=A+ct'$, $x''=a-A+ct''$ and substituting from (19) and (46), we end up with integrals of the form

$$\int_{-1}^{1}f(t')\left[T_n(t')/\sqrt{1-t'^2}\right]dt'$$

in which the $f(t')$ are nothing more than functions appearing in the integrals I_2, I_3 and I_4 defined by (29), (31), and (33). Therefore, the integration of (13) can be carried out immediately, leading to the result given below; it is written in a compact form for all three cases, $Q=0,\pm 1$, using the following simple notation:

$$Q_N = 1 \quad \text{for } Q=0;\ Q_N = Q(-1)^N \quad \text{for } Q=\pm 1 \quad (56)$$

$$\psi_k(x,y) = \phi_k(x,y;0;A) \quad \text{for } Q=0 \quad (57)$$

$$\psi_k(x,y) = \phi_k(x,y;0;A) + \phi_k(x,y;Q;a-A) \quad \text{for } Q=\pm 1 \quad (58)$$

where

$\phi_k(x, y; l; A)$

$$
\begin{aligned}
&= \frac{c}{\pi(\epsilon_1+\epsilon_2)} \sum_{N=0}^{\infty} Q_N a_N \Bigg[-\frac{1}{2} I_2(N; x-A, -c, y) + \frac{1}{2} I_2(N; x+A-2a, c, y) \\
&+ \frac{1}{2} I_2(N; x+A, c, y) - \frac{1}{2ab_k} [(a-x)(b_k-y) I_2(N; 2a-A, -c, 0) + x(b_k-y) I_2(N; a+A, c, 0) \\
&+ (a-x) y I_2(N; 2a-A, -c, b_k) + xy I_2(N; a+A, c, b_k)] - \frac{(-1)^k}{\epsilon_k} \sum_{m=1}^{\infty} \frac{\sin\left(\frac{m\pi x}{a}\right) \sinh\left(\frac{m\pi y}{a}\right)}{m\pi} \\
&\cdot D_{mNk} + \sum_{m=1}^{\infty} \frac{\sin\left(\frac{m\pi y}{b_k}\right) \sinh\left(\frac{m\pi x}{b_k}\right)}{m\pi \sinh\left(\frac{m\pi a}{b_k}\right)} I_4\left(N, m; -\pi\frac{a+A}{b_k}, -\frac{\pi c}{b_k}\right) \\
&+ \sum_{m=1}^{\infty} \frac{\sin\left(\frac{m\pi y}{b_k}\right) \sinh\left(m\pi \frac{a-x}{b_k}\right)}{m\pi \sinh\left(\frac{m\pi a}{b_k}\right)} I_4\left(N, m; \pi\frac{A-2a}{b_k}, \frac{\pi c}{b_k}\right) \\
&+ \sum_{m=1}^{\infty} \left\{ \sin\left(\frac{m\pi x}{a}\right) \sinh\left(m\pi \frac{b_k - y}{a}\right) \Big/ \left[m\pi B_m \sinh\left(\frac{m\pi}{a} b_k\right)\right]\right\} \\
&\qquad \cdot \sum_{l=1,2} b_{mNl} - (1 - y/b_k)[x g(N; a+A, c) - (x-a) g(N; 2a-A, -c)] \Big/ \left[a\left(\frac{\epsilon_1}{b_1} - \frac{\epsilon_2}{b_2}\right)\right] \\
&+ 2b_1^2 \sum_{n=1}^{\infty} \sin\left(u_n \frac{b_k - y}{b_1}\right) \left[\sinh\left(\frac{u_n x}{b_1}\right) g(N; a+A, c) \right. \\
&\left. + \sinh\left(u_n \frac{a-x}{b_1}\right) g(N; 2a-A, -c)\right] \Big/ \left[H_n u_n^2 \sin\left(\frac{u_n b_k}{b_1}\right)\right] + \sum_{l=1,2} (-1)^l \epsilon_l \\
&\cdot \sum_{q=1}^{\infty} \left\{ \frac{\sin\left(\frac{\pi q y}{b_l}\right) \delta\left(\frac{b_k}{b_l} q\right) \left[\sinh\left(\pi q \frac{a-x}{b_l}\right) I_4\left(N, q; \pi\frac{A-2a}{b_l}, \frac{\pi c}{b_l}\right) + \sinh\left(\frac{\pi q x}{b_l}\right) I_4\left(N, q; -\pi\frac{a+A}{b_l}, -\frac{\pi c}{b_l}\right)\right]}{q\pi b_k \sinh\left(\frac{\pi q a}{b_l}\right) \left[\frac{\epsilon_1}{b_1} \delta\left(\frac{b_1}{b_l} q\right) - \frac{\epsilon_2}{b_2} \delta\left(\frac{b_2}{b_l} q\right)\right]} \right. \\
&- \frac{2}{b_l} \sum_{n=1}^{\infty} \sin\left(u_n \frac{b_k - y}{b_1}\right) \left[\sinh\left(u_n \frac{a-x}{b_1}\right) I_4\left(N, q; \pi\frac{A-2a}{b_l}, \frac{\pi c}{b_l}\right) + \sinh\left(u_n \frac{x}{b_1}\right) \right. \\
&\left.\left. \cdot I_4\left(N, q; -\pi\frac{a+A}{b_l}, -\frac{\pi c}{b_l}\right)\right] \Big/ \left\{ H_n \sin\left(u_n \frac{b_k}{b_1}\right) \left[\left(\frac{q\pi}{b_l}\right)^2 - \left(\frac{u_n}{b_1}\right)^2 \right] \right\} \right\} \Bigg]. \qquad (59)
\end{aligned}
$$

The field components $E_x = -\partial\psi/\partial x$, $E_y = -\partial\psi/\partial y$ are obtainable from (57)–(59) by direct differentiation.

TABLE I

$a_0 \times 10^9$	$a_1 \times 10^{10}$	$a_2 \times 10^{10}$	$a_3 \times 10^{10}$	$a_4 \times 10^{11}$	$a_5 \times 10^{11}$	$a_6 \times 10^{12}$
			Q=0			
.16503972	.69760991					
.16580063	.71207912	.24783549				
.16589042	.71363888	.24902300	.10272891			
.16589882	.71380613	.24916498	.10282790	.36188051		
.16589969	.71382505	.24918296	.10284161	.36196967	.13021395	
.16589979	.71382739	.24918536	.10284361	.36198374	.13022276	.4961495
			Q=1			
$a_0 \times 10^9$	$a_1 \times 10^{10}$	$a_2 \times 10^{10}$	$a_3 \times 10^{10}$	$a_4 \times 10^{11}$	$a_5 \times 10^{11}$	$a_6 \times 10^{12}$
.15221197	.68273597					
.15267859	.69523549	.22274080				
.15274041	.69658409	.22384067	.97737300			
.15274589	.69672972	.22396869	.97827682	.33541122		
.15274645	.69674623	.22398514	.97840342	.33549406	.12181764	
.15274652	.69674827	.22398734	.9784218	.33550713	.12182584	.46311644
			Q=-1			
$a_0 \times 10^9$	$a_1 \times 10^{10}$	$a_2 \times 10^{10}$	$a_3 \times 10^{10}$	$a_4 \times 10^{11}$	$a_5 \times 10^{11}$	$a_6 \times 10^{12}$
.17872877	.70775257					
.17987161	.72441468	.27554206				
.17994227	.72620774	.27682951	.10762585			
.18000945	.72639918	.27698752	.10773426	.38995803		
,18001072	.72642079	.27700723	.10774910	.39005391	.13894181	
.18001088	.72642347	.17700987	.10775126	.39006906	.13895127	.53070968

$a = 4,\ b_1 = 2,\ b_2 = -1,\ A = 3.4,\ c = 0.5,\ \epsilon_2 = 10\epsilon_1$

V. Numerical Results and Comparisons

We start by providing in Table I results for the expansion coefficients a_N of $\sigma(t)$ in the case $a = 4$, $b_1 = 2$, $b_2 = -1$, $A = 3.4$, $c = 0.5$, $\epsilon_2 = 10\epsilon_1$, and $Q = 0$. The strip is placed very close to the right wall $x = a$ and the results show that by retaining the first seven coefficients an accuracy in $\sigma(t)$ and $\psi_k(x, y)$ of at least three significant decimals is obtained, i.e., $|a_6/a_0| \cong 0.001$. In other situations, with the strip closer to the center, even fewer a_N will suffice. By providing values of a_N, in C/m^2, for successive values of the truncation size N_m, Table I shows, further, how rapidly the a_N settle to their "final" values and with what accuracy. As observed previously [3], the evaluations of the characteristic impedance Z_0 of the shielded microstrip and of its ϵ_{eff} depend only on a_0 (as well as on a_0 for $\epsilon_2 = \epsilon_1$, a special case of this paper's analysis that may be based on the simpler forms G_α, G_β of G obtained from [1]), which settles to its "final" value for much smaller truncation sizes ($N_m = 3$ provides four-decimal accuracy for a_0); however, values of $\sigma(t)$ or of the field $\psi_k(x, y)$ require the use of more a_N.

In Table II and for $a = 4$, $b_1 = 2$, $b_2 = -1$, and $\epsilon_2 = 10\epsilon_1$ we provide computed values of Z_0 and ϵ_{eff} for various positions A and widths $2c$ of the strip conductors in all three cases $Q = 0, \pm 1$. Finally, in Table III, Z_0 is computed and compared with values obtained previously [7] by two or three other numerical methods. It is remarkable to notice that our values of Z_0, being in very satisfactory agreement with those of [7], are obtained, with the indicated accuracy, even for truncation size $N_m = 0$; sizes $N_m = 2$ and higher do not change them. This is a further

TABLE II

A	c	Q	$Z_0(\Omega)$	ϵ_{eff}
2.4	0.1	0	85.63109	5.80037
		1	103.52725	5.97238
		-1	68.10269	5.54730
	0.2	0	67.99315	5.87613
		1	87.23439	6.06005
		-1	50.43838	5.56201
	0.3	0	57.77971	5.93601
		1	79.66423	6.10761
		-1	40.44088	5.57232
	0.35	0	53.93730	5.96210
		1	77.98133	6.11652
		-1	37.16603	5.57302
2.9	0.1	0	83.65437	5.73532
		1	87.88445	5.85348
		-1	79.46774	5.60811
	0.2	0	65.96206	5.79457
		1	70.41284	5.94135
		-1	61.71794	5.63480
	0.3	0	55.65141	5.84002
		1	60.49404	6.01053
		-1	51.31100	5.67509
	0.4	0	48.36002	5.87600
		1	53.79658	6.06690
		-1	43.89081	5.66515
	0.5	0	42.70333	5.90351
		1	48.98706	6.11041
		-1	38.08650	5.67076
3.4	0.2	0	57.96996	5.64666
		1	59.00741	5.69811
		-1	56.97391	5.59594
	0.4	0	38.45592	5.67509
		1	40.34663	5.75084
		-1	36.74728	5.60572

$a = 4,\ b_1 = 2,\ b_2 = -1,\ \epsilon_2 = 10\epsilon_1$

TABLE III
COMPARISON OF Z_0 WITH EXISTING RESULTS

$\lvert b_1/b_2 \rvert$	$a/(2c)$	$\lvert 2c/b_2 \rvert$	ϵ_2/ϵ_1	$Z_0(\Omega)$	Values of $Z_0(\Omega)$ from [7]		
5	10	1	9.6	49.08	48.4	48.5	
9	10	2	9.6	33.63	33.1	32.8	
5	10	0.8	9.6	54.45	53.8	53.9	
5	6	1	9.6	48.60	47.9	48.5	
5	10	0.4	9.6	70.38	69.7	70.9	
9	10	1	6.0	61.24	60.49	62.71	60.97
9	10	4	6.0	26.09	25.95	27.30	26.03
9	10	0.4	6.0	87.41	86.30	91.37	89.91

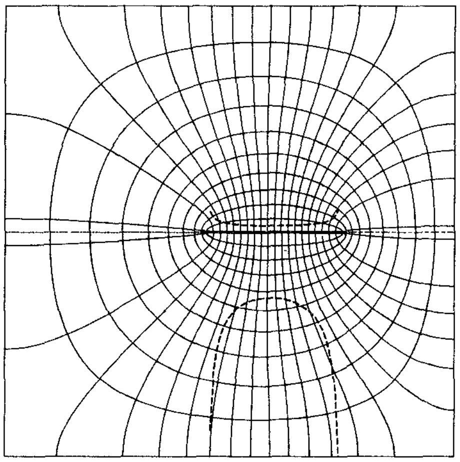


Fig. 3. Surface charge density $\sigma(x)$ and equipotential and field lines of shielded single microstrip: $a=2$, $b_1=-b_2=1$, $A=1.2$, $c=0.3$, $\epsilon_2=10\epsilon_1$, $Q=0$.

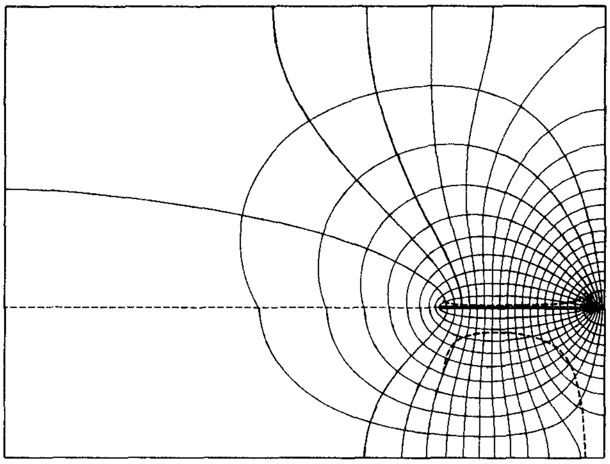


Fig. 4. Surface charge density $\sigma(x)$ and equipotential and field lines of single microstrip very near the shield: $a=4$, $b_1=2$, $b_2=-1$, $A=3.4$, $c=0.5$, $\epsilon_2=10\epsilon_1$, $Q=0$.

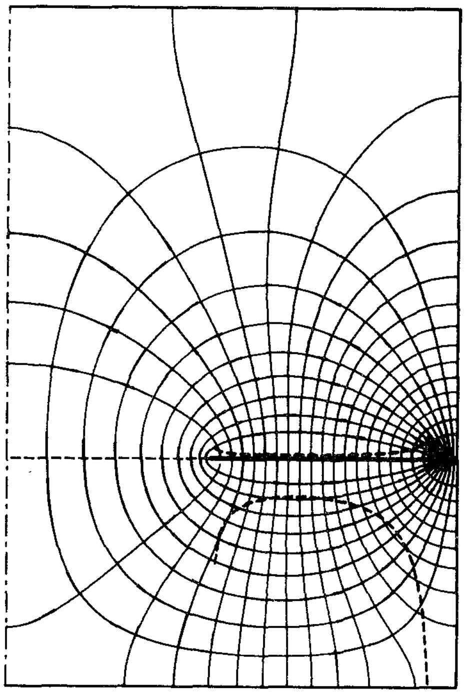


Fig. 5. Same as Fig. 4, but for double microstrip with $Q=-1$. Only the right half-section $a/2 \leqslant x \leqslant a$ is shown.

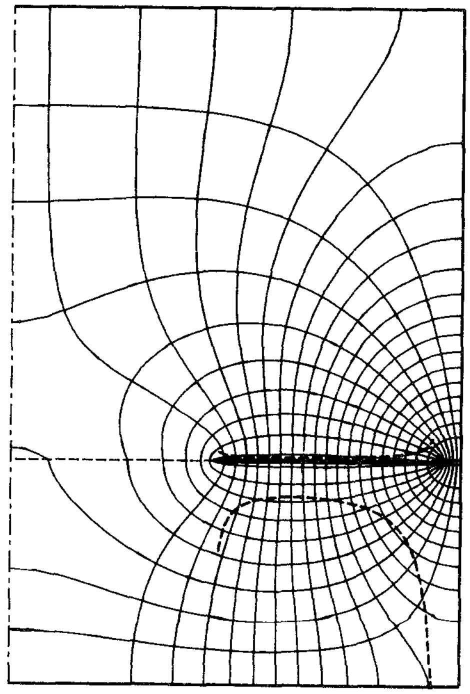


Fig. 6. Same as Fig. 5, but with $Q=1$.

corroboration of the previous remarks and is due, of course, to the fact that the strip is rather narrow and quite far from the walls.

What is claimed herein is that, beyond its generality, our exact analytical treatment is applicable to cases of close proximity of the strip to the walls or between the strips (for $Q=\pm 1$) and to very wide strips as well. In addition, it is able to provide accurate results for the field everywhere and in all cases by using the appropriate form of G. This aspect has been discussed in detail previously [1]. Here we provide, in Figs. 3–6, detailed field plots: the nine equipotentials $\psi=0.9$–0.8–$\cdots$–0.1 between $\psi=1$ on the strip and $\psi=0$ on the shield, as well as the 20 field lines that cut the $2c$-wide strip at equispaced intervals of $c/10$. A dotted line plot of $\sigma^+(t)$, $\sigma^-(t)$, on the upper and lower faces of the strip, is also drawn. The cases $Q=0$, with a certain shield width a, cover also configurations with $Q=-1$ and double shield width $2a$. In Figs. 5 and 6, for $Q=-1$ and $Q=1$, only the right half $(a/2 \leqslant x \leqslant a)$ part of the plot is given, the left half being its image with respect to the midplane $x=a/2$. A truncation size of $N_m=6$ or 7 was used in all these plots, most of which, as may be observed, allow very close proximity of the strip to the shield. In all cases $\epsilon_2=10\epsilon_1$ was considered. For some configurations comparisons were also made with the case $\epsilon_2=\epsilon_1$. No plots are included here, but the main difference they exhibited was the expected concentration of the equipotential lines near the strip when ϵ_2 changed from ϵ_1 to $10\epsilon_1$.

APPENDIX

Here we outline the analytical evaluation of the integrals I, I_1 to I_6, P_{MN}, and W_{MN} defined in (24), (26), (29), (31), (33), (36), (39), (52), and (53). Starting with (24), we use the change of variables $t = \cos\theta$ and $\tau = \cos\varphi$ and reverse the order of integration to obtain

$$I = \int_0^\pi \sin^2\varphi \int_0^\pi \frac{\cos(M\theta)}{\cos\varphi - \cos\theta} d\theta\, d\varphi$$

$$= -\pi \int_0^\pi \sin\varphi \sin M\varphi\, d\varphi = -\frac{\pi^2}{2}\delta_{M1}. \quad \text{(A1)}$$

In obtaining this result use was made of the basic Hilbert formula [8, p. 207]:

$$\int_0^\pi \frac{\cos(M\theta)}{\cos\theta - \cos\varphi} d\theta = \int_{-1}^1 \frac{T_M(t)}{\sqrt{1-t^2}} \frac{dt}{t-\tau} = \pi \frac{\sin(M\varphi)}{\sin\varphi},$$
$$\tau = \cos\varphi, t = \cos\theta, M = 0,1,2,\cdots. \quad \text{(A2)}$$

Next come the integrals I_1, I_5 defined in (26), (36). With t', τ in $[-1,1]$ and the restriction $|z_2/z_1| < 1/2$ we obtain

$$\ln|z_1 + z_2(\tau + t')|$$
$$= \ln|z_1| - \sum_{p=1}^{\infty} \frac{1}{p}\left(-\frac{z_2}{z_1}\right)^p (\tau + t')^p$$
$$= \ln|z_1| - \sum_{p=1}^{\infty} \frac{1}{p}\left(-\frac{z_2}{z_1}\right)^p \sum_{s=0}^{p} \binom{p}{s} \tau^s t'^{p-s} \quad \text{(A3)}$$

$$\frac{1}{z_1 + z_2(\tau + t')} = \frac{1}{z_1}\sum_{p=0}^{\infty}\left(-\frac{z_2}{z_1}\right)^p (\tau + t')^p$$
$$= \frac{1}{z_1}\sum_{p=0}^{\infty}\left(-\frac{z_2}{z_1}\right)^p \sum_{s=0}^{p}\binom{p}{s}\tau^s t'^{p-s}. \quad \text{(A4)}$$

We consider now the following integral (see, also, [9, p. 374]):

$$J(s,N) = \int_{-1}^1 t^s \frac{T_N(t)}{\sqrt{1-t^2}} dt = \int_0^\pi \cos^s\theta \cos N\theta\, d\theta$$
$$= \begin{cases} 0, & s < N \\ \dfrac{\pi}{2^s}\begin{pmatrix} s \\ \dfrac{s+N}{2}\end{pmatrix}, & s \geqslant N \text{ and } s+N \text{ even} \\ 0, & s \geqslant N \text{ and } s+N \text{ odd.}\end{cases} \quad \text{(A5)}$$

Substituting (A3) into (26) and using (A5) we obtain

$$I_1(N; z_1, z_2) = \ln|z_1|\pi^2\delta_{N0} - \sum_{p=\max(1,N)}^{\infty} \frac{1}{p}\left(-\frac{z_2}{z_1}\right)^p$$
$$\cdot \sum_{s=0}^{p-N}\binom{p}{s} J(s,0) J(p-s,N) \quad \text{(A6)}$$

while from (A4), (36), (A2), and the changes of variable $t' = \cos\theta$, $\tau = \cos\varphi$,

$$I_5 = -\pi\int_0^\pi \cos N\theta \int_0^\pi \sin\varphi \sin(M\varphi)\frac{1}{z_1}\sum_{p=0}^{\infty}\left(-\frac{z_2}{z_1}\right)^p$$
$$\cdot \sum_{s=0}^{p}\binom{p}{s}\cos^s\varphi \cos^{p-s}\theta\, d\theta\, d\varphi.$$

Using $2\sin\varphi\sin(M\varphi) = \cos(M-1)\varphi - \cos(M+1)\varphi$ we finally get

$$I_5(N, M; z_1, z_2) = -\frac{\pi}{2z_1}\sum_{p=N}^{\infty}\left(-\frac{z_2}{z_1}\right)^p$$
$$\sum_{s=\max(0,M-1)}^{p-N}\binom{p}{s} J(p-s,N)$$
$$\cdot [J(s, M-1) - J(s, M+1)]. \quad \text{(A7)}$$

Coming next to I_2 in (29) and observing that with real z_1, z_2, z_3,

$$\ln\left[(z_1 + z_2 t')^2 + z_3^2\right] = 2\,\mathrm{Re}\ln(z_1 + iz_3 + z_2 t')$$

we obtain

$$I_2(N; z_1, z_2, z_3) = 2\,\mathrm{Re}\int_{-1}^1 \ln(z_1 + iz_3 + z_2 t)\frac{T_N(t)}{\sqrt{1-t^2}} dt$$
$$= 2\,\mathrm{Re}\int_0^\pi \ln(z_1 + iz_3 + z_2\cos\theta)\cos N\theta\, d\theta$$
$$= \mathrm{Re}\int_{-\pi}^\pi \ln(z_1 + iz_3 + z_2\cos\theta)\cos N\theta\, d\theta. \quad \text{(A8)}$$

For $N > 0$ we integrate by parts:

$$I_2 = \frac{1}{N}\mathrm{Re}\int_{-\pi}^{\pi}\frac{z_2 \sin N\theta \sin\theta}{z_1 + iz_3 + z_2\cos\theta} d\theta$$
$$= -\frac{z_2}{N}\mathrm{Re}\int_{-\pi}^{\pi}\frac{ie^{iN\theta}\sin\theta}{z_1 + iz_3 + z_2\cos\theta} d\theta \quad \text{(A9)}$$

and use the change of variable $\zeta = e^{i\theta}$, $d\zeta = i\zeta\, d\theta$:

$$I_2 = \frac{1}{N}\mathrm{Re}\oint_{|\zeta|=1}\frac{\zeta^{N-1}(\zeta^2-1)i}{\zeta^2 + 2\dfrac{z_1 + iz_3}{z_2}\zeta + 1} d\zeta$$
$$= \frac{1}{N}\mathrm{Re}\oint_{|\zeta|=1}\frac{\zeta^{N-1}(\zeta^2-1)i}{(\zeta-\zeta_1)(\zeta-\zeta_2)} d\zeta$$
$$= -\frac{2\pi}{N}\mathrm{Re}\left[\frac{\zeta_1^{N-1}(\zeta_1^2-1)}{\zeta_1-\zeta_2}\right].$$

It is obvious that $\zeta_2 = 1/\zeta_1$; therefore, one of the roots of $\zeta^2 + 2(z_1 + iz_3)\zeta/z_2 + 1 = 0$, say ζ_1, falls inside the unit circle, the other outside. Finally,

$$I_2(N; z_1, z_2, z_3) = -\frac{2\pi}{N}\mathrm{Re}(\zeta_1^N), \qquad N > 0, \quad |\zeta_1| < 1. \quad \text{(A10)}$$

For $N=0$ we get, from (A8),

$$I_2(0;z_1,z_2,z_3)=2\,\mathrm{Re}\int_0^{\pi}\ln(z_1+iz_3+z_2\cos\theta)\,d\theta$$
$$=2\pi\ln\left|\frac{z_1+iz_3+\sqrt{(z_1+iz_3)^2-z_2^2}}{2}\right|. \quad \text{(A11)}$$

This result was obtained on the basis of the standard integral [9, p. 527]

$$\int_0^{\pi}\ln(a+b\cos x)\,dx=\pi\ln\frac{\alpha+\sqrt{a^2-b^2}}{2}, \qquad 0<|b|\leqslant a. \quad \text{(A12)}$$

Here we have $|z_2|<|z_1+iz_3|$, but $a=z_1+iz_3$ is now complex. However, it is easily shown that the formula holds for complex a as well.

For the integral I_3 in (31) we change the variable $t'=\cos\theta$ to get

$$I_3(N;z_1,z_2)=\int_0^{\pi}\cos N\theta\sin(z_1+z_2\cos\theta)\,d\theta$$
$$=\sin z_1\int_0^{\pi}\cos N\theta\cos(z_2\cos\theta)\,d\theta+\cos z_1\int_0^{\pi}\cos N\theta\sin(z_2\cos\theta)\,d\theta$$
$$=\begin{cases}\pi\sin z_1(-1)^{N/2}J_N(z_2), & N \text{ even}\\ -\pi\cos z_1(-1)^{(N+1)/2}J_N(z_2), & N\text{ odd}\end{cases} \quad \text{(A13)}$$

where $J_N(z_2)$ is the Bessel function and the final result was based on standard integrals [4, p. 361].

We next go to I_6 and perform first the principal value integration with respect to t. Changing the variables $t=\cos\theta$, $\tau=\cos\varphi$ and using (A2) we obtain

$$I_6=\int_0^{\pi}\sin^2\varphi\cos(z_3+z_4\cos\varphi)\int_0^{\pi}\frac{\cos(M\theta)\,d\theta}{\cos\varphi-\cos\theta}\,d\varphi$$
$$=-\pi\int_0^{\pi}\sin\varphi\cos(z_3+z_4\cos\varphi)\sin M\varphi\,d\varphi$$
$$=-\frac{\pi}{2}\int_0^{\pi}[\cos z_3\cos(z_4\cos\varphi)-\sin z_3\sin(z_4\cos\varphi)]\cdot[\cos(M-1)\varphi-\cos(M+1)\varphi]\,d\varphi$$
$$=-\frac{\pi^2}{2}\left\{\cos z_3\sin\frac{M\pi}{2}[J_{M-1}(z_4)+J_{M+1}(z_4)]+\sin z_3\cos\frac{M\pi}{2}[J_{M-1}(z_4)+Z_{M+1}(z_4)]\right\}.$$

In the above use of standard integrals from [9, p. 402] was made. Finally, even for $M=0$, we obtain

$$I_6(M;z_3,z_4)=-\frac{M\pi^2}{z_4}J_M(z_4)\sin\left(\frac{M\pi}{2}+z_3\right). \quad \text{(A14)}$$

Next comes the integral I_4 in (33); $d_M(z)$ is defined in (7) and it is obvious that I_4 is the sum of four integrals:

$$I_4(N,m;z_1,z_2)=(-1)^mI_N(mz_1-im\pi,mz_2)-I_N(mz_1,mz_2)+(-1)^mI_N(-m\bar z_1+im\pi,-m\bar z_2)-I_N(-m\bar z_1,-m\bar z_2) \quad \text{(A15)}$$

where

$$I_N(z_3,z_4)=\mathrm{Re}\int_{-1}^{1}\exp(z_3+z_4t)E_1(z_3+z_4t)\frac{T_N(t)}{\sqrt{1-t^2}}\,dt \quad \text{(A16)}$$

z_3, z_4 being, in general, complex constants. Changing the variable $t=\cos\theta$ in (A17), integrating by parts, and using the following relation [4]: $d/dz[e^zE_1(z)]=e^zE_1(z)-1/z$, we obtain

$$I_N(z_3,z_4)=\mathrm{Re}\int_0^{\pi}e^{z_3+z_4\cos\theta}E_1(z_3+z_4\cos\theta)\cos N\theta\,d\theta$$
$$=\mathrm{Re}\,\frac{z_4}{N}\int_0^{\pi}\sin N\theta\sin\theta\cdot\left[e^{z_3+z_4\cos\theta}E_1(z_3+z_4\cos\theta)-1/(z_3+z_4\cos\theta)\right]d\theta. \quad \text{(A17)}$$

For $N>0$ we use $\sin N\theta\sin\theta=\frac{1}{2}\cos(N-1)\theta-\frac{1}{2}\cos(N+1)\theta$ and following the steps that led from (A9) to (A10) we obtain the recurrence relation:

$$I_N(z_3,z_4)=\frac{z_4}{2N}[I_{N-1}(z_3,z_4)-I_{N+1}(z_3,z_4)]+\frac{\pi}{N}\mathrm{Re}(\zeta_1^N), \quad N>0 \quad \text{(A18)}$$

where ζ_1 is the root of $\zeta^2+2(z_3/z_4)\zeta+1=0$ with $|\zeta_1|<1$. To use (A18) requires the evaluation of $I_N(z_3,z_4)$ for two successive values of N. Although it is possible to obtain an analytic evaluation of $I_N(z_3,z_4)$—on the basis of the Maclaurin's expansion of $E_1(z)+\ln z$ and the use of the integrals $J(s,N)$, in (A5), and $I_2(N;z_1,z_2,0)$—the resulting expressions, involving triple series, are cumbersome numerically. It was found preferable in this particular case to evaluate numerically $I_N(z_3,z_4)$ for either $N=1$, 2 or $N=6$, 7 and to use the recurrence formula (A18) forward or backward, respectively. As has been discussed above in Section V $N=7$ is sufficient as an upper matrix size. It was also noted that (A18) works better in the backward direction, although with N limited up to 7 no problem arises in either direction.

There remain the integrals P_{MN}, W_{MN} defined in (52), (53), occurring in double-strip configurations. They are very similar to the integrals I_1, I_5 in (26), (36), differing only in that the sum $\tau+t'$ in the latter is replaced by $\tau-t'$. Following the steps indicated in (A3)–(A7), we end up with

$$P_{MN} = -\frac{\delta_{M0}}{\pi \ln 2} \sum_{p=\max(1,N)}^{\infty} \frac{1}{p}\left(\frac{c}{2A-a}\right)^{N} \cdot \sum_{s=0}^{p-N} (-1)^s \binom{p}{s} J(s,0) J(p-s,N) \quad \text{(A19)}$$

$$W_{MN} = -\frac{1}{2\pi} \sum_{p=N}^{\infty} \left(\frac{c}{2A-a}\right)^{p+1} \cdot \sum_{s=\max(0,M-1)}^{p-N} (-1)^s \binom{p}{s} J(p-s,N) \cdot [J(s,M-1) - J(s,M+1)]. \quad \text{(A20)}$$

References

[1] J. G. Fikioris, J. L. Tsalamengas, and G. J. Fikioris, "Strongly convergent Green's function expansions for rectangularly shielded microstrip lines," *IEEE Trans. Microwave Theory Tech.*, vol. 36, pp. 1386–1396, Oct. 1988.

[2] J. G. Fikioris and J. L. Tsalamengas, "Strongly and uniformly convergent Green's function expansions," *J. Franklin Inst.*, vol. 324, no. 1, pp. 1–17, 1987.

[3] J. G. Fikioris and J. L. Tsalamengas, "Exact solutions for rectangularly shielded lines by the Carleman–Vekua method," *IEEE Trans. Microwave Theory Tech.*, vol. 36, pp. 659–675, Apr. 1988.

[4] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*. New York: Dover, 1972.

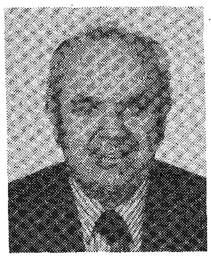
[5] G. F. Carrier, M. Krook, and C. E. Pearson, *Functions of a Complex Variable*. New York: McGraw-Hill, 1966.

[6] F. D. Gakhov, *Boundary Value Problems*. Oxford: Pergamon Press, 1966 (English translation by I. N. Sneddon).

[7] H.-Y. Yee and K. Wu, "Printed circuit transmission-line characteristic impedance by transverse modal analysis," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-34, pp. 1157–1163, Nov. 1986.

[8] R. P. Kanwal, *Linear Integral Equations, Theory and Technique*. New York: Academic Press, 1971.

[9] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products*, 4th ed. New York: Academic Press, 1965 (English translation by A. Jeffrey).



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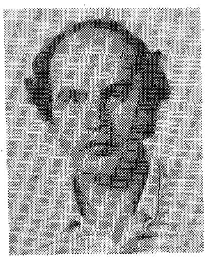

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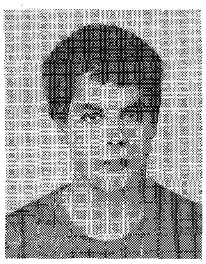

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